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Radiation Physics Group

INTERNAL REPORT

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SHAPING ELECTRON BEAMS WITH QUADRUPOLE MAGNETS

by W. T. Link

This report examines theoretically the possibility of obtaining electron beams of elliptical cross section by passing electron beams of circular cross section through quadrupole magnets. Several basic concepts must be introduced.

1. Beam Rays

Consider a beam of electrons to consist of many separate rays. Then each ray is followed separately through a system of magnets and drift spaces. The ray is completely specified at any distance along its trajectory by its horizontal and vertical distances x and y from the center line, by its horizontal and vertical slopes x' and y' with respect to the center line, and by the momentum p of the electrons in the ray.

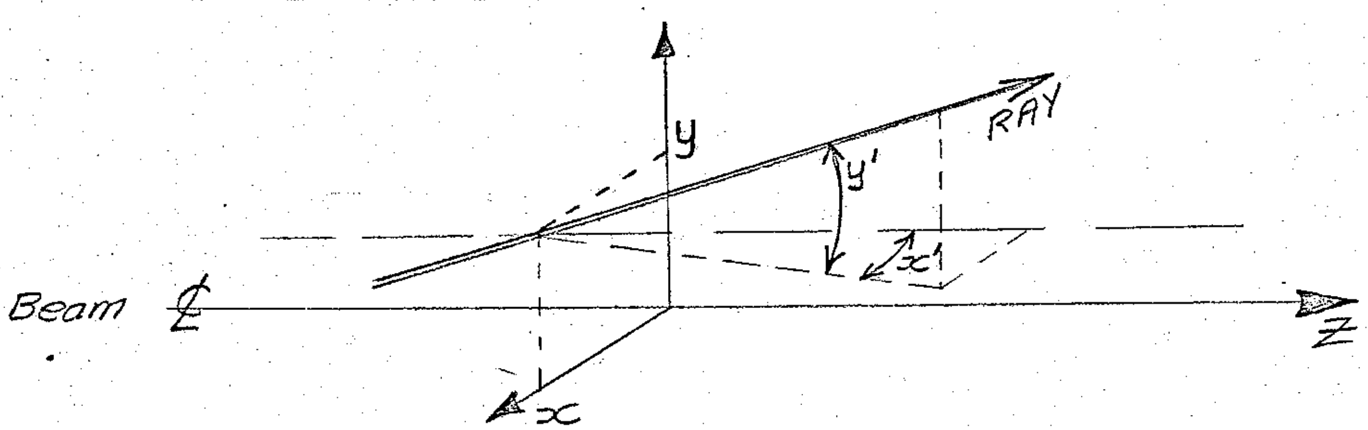


FIGURE 1. A single beam ray. x, x', y, y', are all positive in the example.

2. Phase Space and Phase Space Area

If at a given z each ray in a beam is measured and its position is plotted against its slope, the result is a phase space plot.

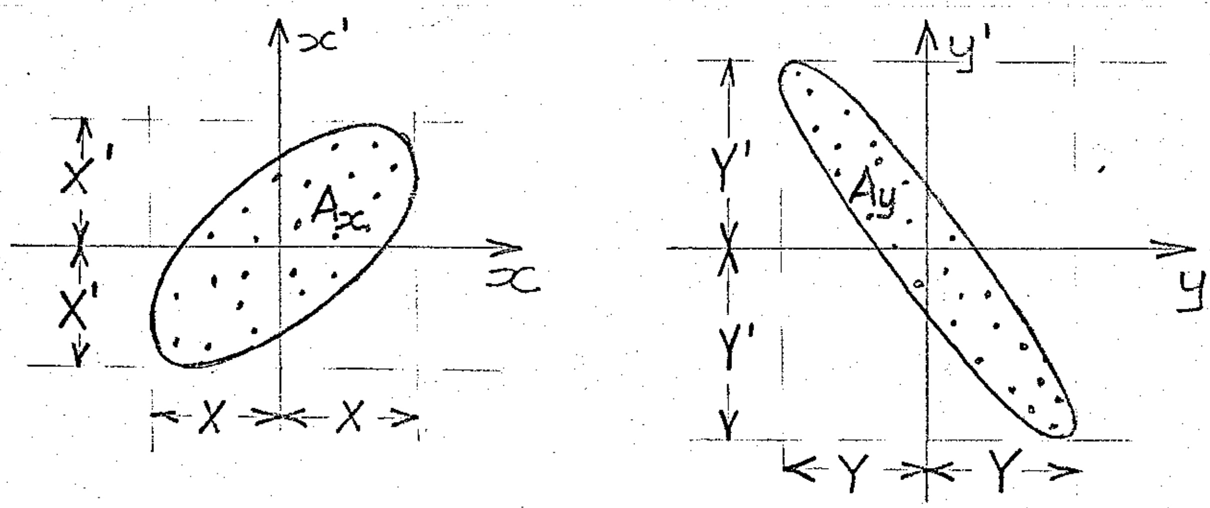


FIGURE 2. Phase Space Plots

One must specify both horizontal and vertical positions and slopes, and horizontal and vertical phase space plots result. The points in these phase space plots often lie within roughly elliptical boundaries and this is assumed in Figure 2.

The ellipses in Figure 2 are specified by the parametric equations

$$x = X \cos(\theta_x + \psi_x) \qquad y = Y \cos(\theta_y + \psi_y)$$

$$x' = X' \sin(\theta_x - \psi_x) \qquad y' = Y' \sin(\theta_y - \psi_y)$$

$$A_x = \pi X X' \cos 2\psi_x \qquad A_y = \pi Y Y' \cos 2\psi_y .$$

Here θ_x, θ_y are parameters which when varied cause the equations to trace out the ellipses. ψ_x and ψ_y are constants which determine the slopes of the ellipses, and X, X', Y, Y' as shown in Figure 2 are the half sizes of the ellipses. In most of the low current density beam transport theory found in the literature, the areas A_x and A_y of the two ellipses (or of whatever shape is appropriate) are assumed constant. Figure 4 illustrates the behavior of a phase space plot as a beam translates through a position of minimum size or a focus. Note the areas of the phase space plots are the same, although X and ψ_x vary widely at different points along the beam.

3. The Transport Matrix

Write the position and slope of a ray as a 2 x 1 matrix

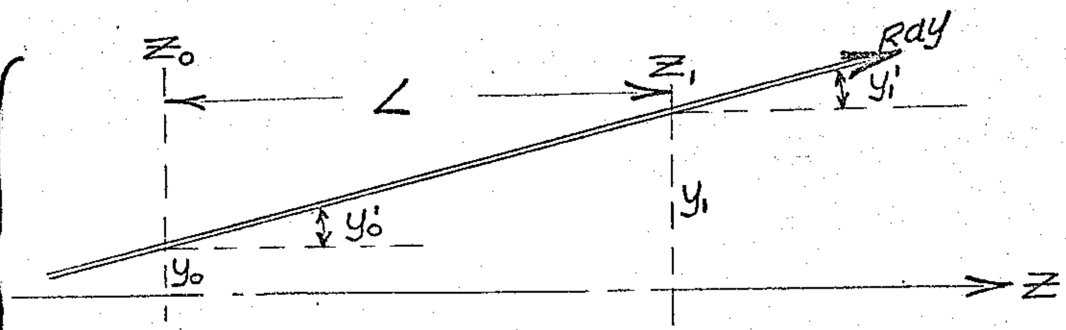
$$\begin{pmatrix} x \\ x' \end{pmatrix}.$$

The matrix representing the ray changes as the ray translates along the z axis. Then quite generally

$$\begin{pmatrix} x_1 \\ x_1' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_o \\ x_o' \end{pmatrix}.$$

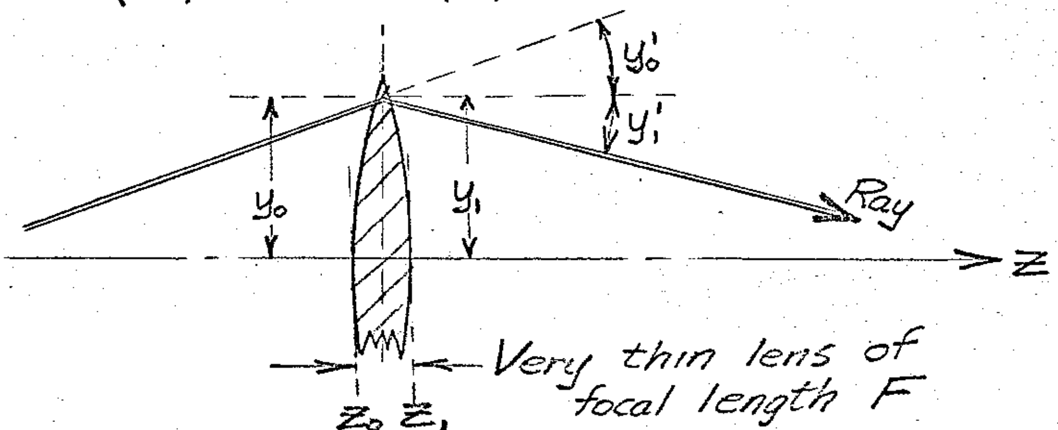
The matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is called a transport matrix and is said to transport the beam from z_o to z_1. The three examples in Figure 3 illustrate this.

Drift Space



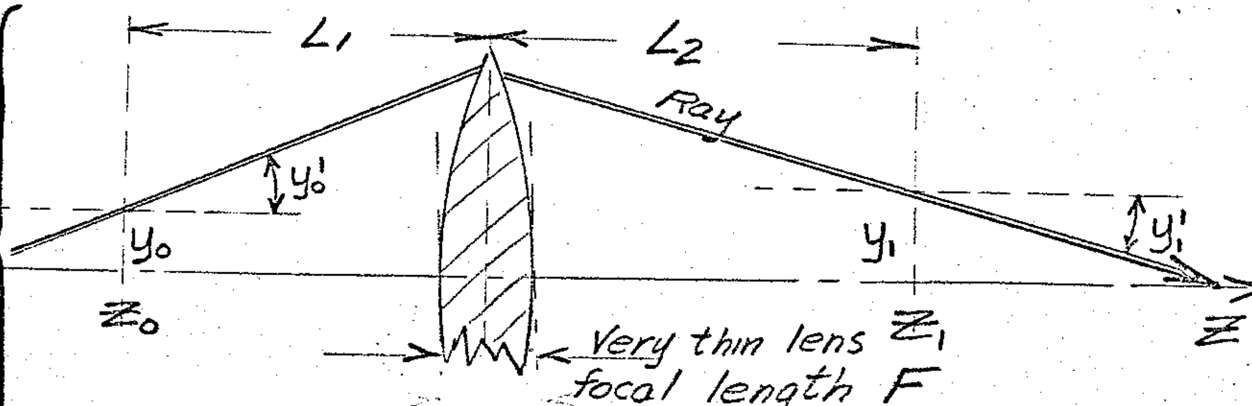
$$\begin{pmatrix} y_1 \\ y_1' \end{pmatrix} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y_0 \\ y_0' \end{pmatrix} \quad \text{i.e.} \quad \begin{aligned} y_1 &= y_0 + L y_0' \\ y_1' &= y_0' \end{aligned}$$

Thin Lens



$$\begin{pmatrix} y_1 \\ y_1' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{F} & 1 \end{pmatrix} \begin{pmatrix} y_0 \\ y_0' \end{pmatrix} \quad \text{i.e.} \quad \begin{aligned} y_1 &= y_0 \\ y_1' &= -\frac{1}{F} y_0 + y_0' \end{aligned}$$

Two Drift Spaces and A Thin Lens



$$\begin{pmatrix} y_1 \\ y_1' \end{pmatrix} = \begin{pmatrix} 1 & L_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{F} & 1 \end{pmatrix} \begin{pmatrix} 1 & L_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y_0 \\ y_0' \end{pmatrix}$$

$$\begin{pmatrix} y_1 \\ y_1' \end{pmatrix} = \begin{pmatrix} 1 - \frac{L_2}{F} & L_1 + L_2 - \frac{L_1 L_2}{F} \\ -\frac{1}{F} & 1 - \frac{L_1}{F} \end{pmatrix} \begin{pmatrix} y_0 \\ y_0' \end{pmatrix}$$

FIGURE 3. Examples of the transport matrix

4. Transporting the Phase Space Ellipse

Assume a phase space ellipse X_o, X_o', ψ_o changes into X_1, X_1', ψ_1 by passing through various transport elements summarized in the single transport matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Then
$$X_1^2 = a^2 X_o^2 + b^2 \left(X_o'\right)^2 - 2\,a\,b\,X_o X_o' \sin 2\psi_o$$

$$\left(X_1'\right)^2 = c^2 X_o^2 + d^2 \left(X_o'\right)^2 - 2\,c\,d\,X_o X_o' \sin 2\psi_o$$

$$X_1 X_1' \sin 2\psi_1 = X_o X_o' \sin 2\psi_o \,(a\,d + b\,c) - X_o^2\, a\,c - \left(X_o'\right)^2 b\,d$$

$$A = \pi X_o X_o' \cos 2\psi_o = \pi X_1 X_1' \cos 2\psi_1$$

5. The Description of a Focus

A beam is said to be focused wherever it passes through a minimum in size. A beam may have different horizontal and vertical focii. A beam of finite phase space area can never have zero size, for then

$$A = X X' \cos 2\psi, \quad X = O, \quad \text{therefore } X' = \infty.$$

Also a beam is focused when $\cos 2\psi = 1$, i.e., when the phase space ellipse is not tilted. Figure 4 illustrates this.

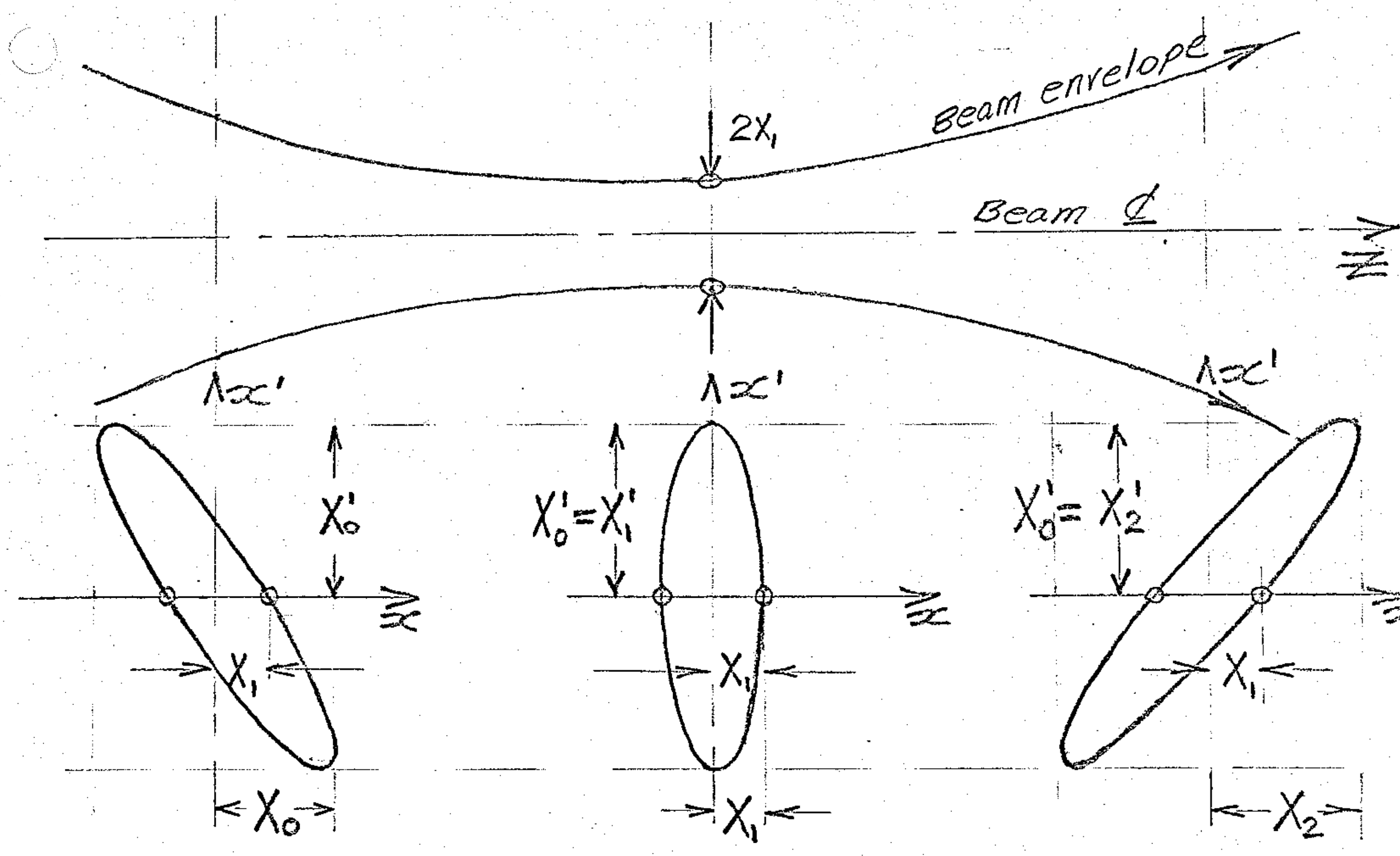


FIGURE 4. A Beam Drifts into a Focus and Out Again. Note that $A_o = A_1 = A_2$ and $X'_o = X'_1 = X'_2$.

6. The Quadrupole Lens

The most flexible and probably the most common beam-focusing device is the quadrupole lens. Beam transport theory must deal separately with the two planes x and y, and the quadrupole lens is a striking example of this, for a quadrupole lens focuses in one plane and defocuses in the other plane. Quadrupole lenses vary in weight from ounces to tons and in aperture from 1/2 in. to 5 ft. The lens is defined magnetically by a gradient k and an effective magnetic length L. Thus $B_x = ky$, $B_y = kx$ and the transport properties of the lens are given by

$$\begin{pmatrix} x_1 \\ x_1' \end{pmatrix} = \begin{pmatrix} \cos \mu L & \frac{1}{\mu} \sin \mu L \\ -\mu \sin \mu L & \cos \mu L \end{pmatrix} \begin{pmatrix} x_o \\ x_o' \end{pmatrix}$$

$$\begin{pmatrix} y_1 \\ y_1' \end{pmatrix} = \begin{pmatrix} \cosh \mu L & \frac{1}{\mu} \sinh \mu L \\ \mu \sinh \mu L & \cosh \mu L \end{pmatrix} \begin{pmatrix} y_o \\ y_o' \end{pmatrix}$$

$$\mu = \left(\frac{K}{B\rho}\right)^{1/2}, \qquad B\rho = \frac{mV}{e}.$$

These transport matrices transport $\begin{pmatrix} x_o \\ x_o' \end{pmatrix}$ and $\begin{pmatrix} y_o \\ y_o' \end{pmatrix}$ at the entrance of the quadrupole to $\begin{pmatrix} x_1 \\ x_1' \end{pmatrix}$ and $\begin{pmatrix} y_1 \\ y_1' \end{pmatrix}$ at the exit of the quadrupole. It is useful to consider a thin quadrupole lens in which the length $L \rightarrow O$ and $k \rightarrow \infty$ in such a way that $\frac{kL}{B\rho} \rightarrow \frac{1}{F}$ where F is finite, i.e., F is some focal length.

Then

$$\begin{pmatrix} x_1 \\ x_1' \end{pmatrix} = \begin{pmatrix} 1 & O \\ -\frac{1}{F} & 1 \end{pmatrix} \begin{pmatrix} x_o \\ x_o' \end{pmatrix}$$

$$\begin{pmatrix} y_1 \\ y_1' \end{pmatrix} = \begin{pmatrix} 1 & O \\ \frac{1}{F} & 1 \end{pmatrix} \begin{pmatrix} y_o \\ y_o' \end{pmatrix}$$

thin lens, $F = \frac{B\rho}{kL}$.

Now a few practical considerations. For 3-MeV electrons, $B\rho \simeq 10^4$ gauss cm. $k = \frac{B_R}{R}$, where B_R is the magnetic field at the maximum radius R. Then

$$\frac{1}{F} = \frac{kL}{B\rho} = \frac{B_R L}{10^4 R} \text{ radians per cm.}$$

Now B_R can easily be as large as 10^4 gauss, and reasonable values of L and R are L = 10 cm, R = 2 cm, and $\frac{1}{F} = \frac{10^4}{2} \frac{10}{10^4} = 5$ radians per cm, or F = 1/5 cm. This shows that the extremely short focal length of 1/5 cm is easily obtained and in the following it is assumed that any reasonable focal length F can be obtained simply by adjusting the current in a thin electromagnet quadrupole. A limit on the shortness of focal length is the desirability of preserving some semblance of the paraxial case, and the necessity of obtaining a focus outside the finite length of the quadrupole magnet. Fairly arbitrarily, the largest angular deflection of any ray permitted in this analysis is about 30° (40° in a few cases).

7. Some Remarks

The determinant of any linear transport matrix is unity. This can be an extremely convenient check on calculations.

To completely specify a beam, one must include momentum spread and this forces the theory into three vectors and three by three matrices. Thus if $\frac{\Delta P}{P}$ is the fractional difference of the momentum of any ray from the mean momentum P,

$$\begin{pmatrix} x_1 \\ x_1' \\ \frac{\Delta P}{P} \end{pmatrix} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \begin{pmatrix} x_o \\ x_o' \\ \frac{\Delta P}{P} \end{pmatrix}.$$

One can then write the transfer matrices for bending magnets with suitable edge angles and magnetic field gradients.

Simultaneous focus in both planes can be obtained with quadrupole doublets or triplets.

Some references:

1. Beam Transport, A Selected, Annotated Bibliography, Robert R. Kepple, ANL-6602

2. Steffen, K. G., High Energy Beam Optics, New York: Interscience, 1965.

8. Assumptions

The following assumptions are made for the distortion calculations of the diverging electron beam from the 730 machine:

(a) Assume that phase space area is conserved for the drifting beam mode.

(b) Assume the beam behaves "properly" in a quadrupole.

(c) Assume the electron beam kinetic energy is 3 MeV and that the standard deviation of the kinetic energy distribution is 0.75 MeV (this is a FWHM of 60 per cent). If the focal length at 3 MeV is F, then for the same quadrupole magnet setting, the focal length at 2.25 MeV is (2.25/3)F, and at 3.75 MeV is (3.75/3)F.

(d) Assume the electron beam necks down just outside the anode to a beam diameter of 2 cm with a maximum half angle of divergence of 0.15 radians. Then at the neck of the beam.

$$X_o = Y_o = 1 \text{ cm}$$
$$X'_o = Y'_o = 0.15 \text{ radians}$$
$$\psi_{xo} = \psi_{yo} = 0$$
$$A_x = A_y = \pi X_o X'_o \cos 2\psi = 0.47 \text{ cm radians}$$

Measure downstream distance z from this neck in the beam. (This simplifies calculations).

9. Calculation of Distortions to Circular Electron Beams

(a) Simplified Phase Space Shape.

It is sometimes a simplification to assume the following parallelogram shape in phase space.

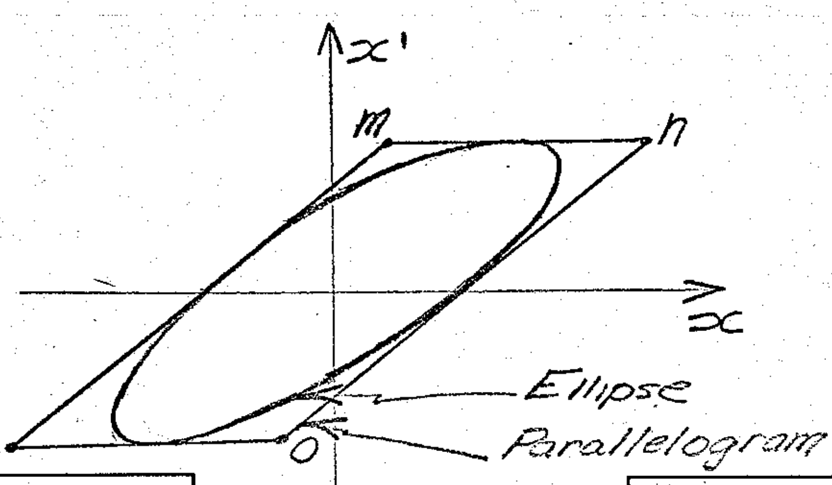


FIGURE 2. The parallelogram is a useful approximation to the ellipse.

It is then only necessary to follow the four points m, n, o, p, through any linear transport system and this can be done quickly with very simple algebra or with graphical methods. Beam sizes so obtained will always be larger than beam sizes obtained from the enclosed elliptical phase space shape. The results of such ray tracing of the four corners of a parallelogram are shown in Figures 7 - 12 inclusive.

Here the beam is assumed to drift in free space from the beam neck near the anode to the quadrupole lens and Figure 6 shows this for the three distances Z = 5, 10, 30 cm.

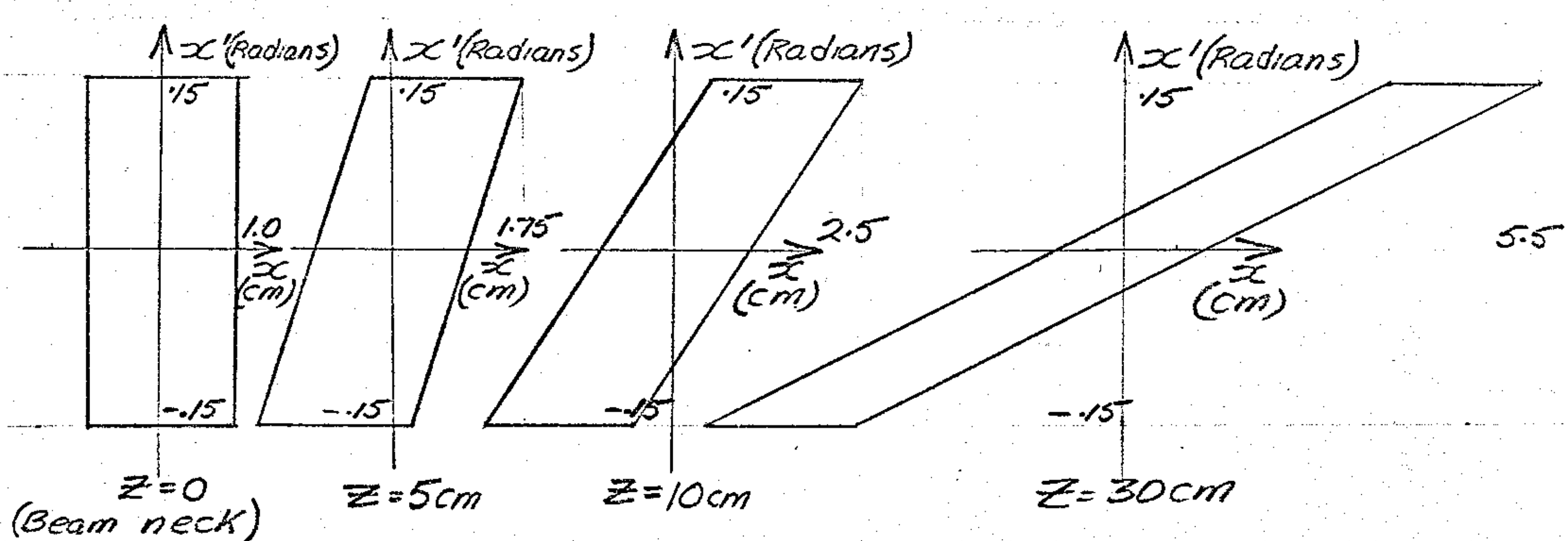


FIGURE 6. Beam drifting from beam waist to quadrupole located at Z = 5, 10, 30 cm.

The quadrupole magnet is then assumed to be adjusted to give a maximum 30° deflection and the four rays are drawn in using the lens formula

$$\Delta \theta = \frac{1}{F} r$$

where $\Delta\theta$ is the angular deflection given by a quadrupole of local length F to a ray at radius r in the quadrupole.

The ray diagrams for the three energies 2.25 MeV, 3.00 MeV, and 3.75 MeV are included and are obtained from the lens formula by using

$$F(E) = \frac{E}{3.00} F(3.00).$$

The resulting envelope of the three ray systems is the over-all envelope of our beam, or at least is an envelope which will contain well over 50 per cent of the beam. The resulting elliptical cross-sections are sketched in Figures 10, 11, and 12.

(b) Elliptical Phase Space Shape

Figures 13 and 14 show the results of transporting one ellipse with three different kinetic energies through a quadrupole lens 30 cm from the beam neck. Comparison of Figures 13 and 9 show close similarity for the ellipse and parallelogram cases and as expected beam cross-sections for the elliptical case are smaller.

10. Conclusions

With reference to Figures 10, 11, 12, and 14, it is obvious that distortion parameters of 8 are obtainable. (Distortion parameter $= \frac{\text{major axis}}{\text{minor axis}}$.)

To obtain larger distortion parameters one or more of the following must be tried.

(a) Reduce kinetic energy spread of the electron beam,

(b) Use quadrupoles of higher power,

(c) Pass the highly distorted electron beam into a pinch chamber,

It is the author's belief (or hope) that the ellipse will then "clap" down into a fine line. This is shown in Figure 15.

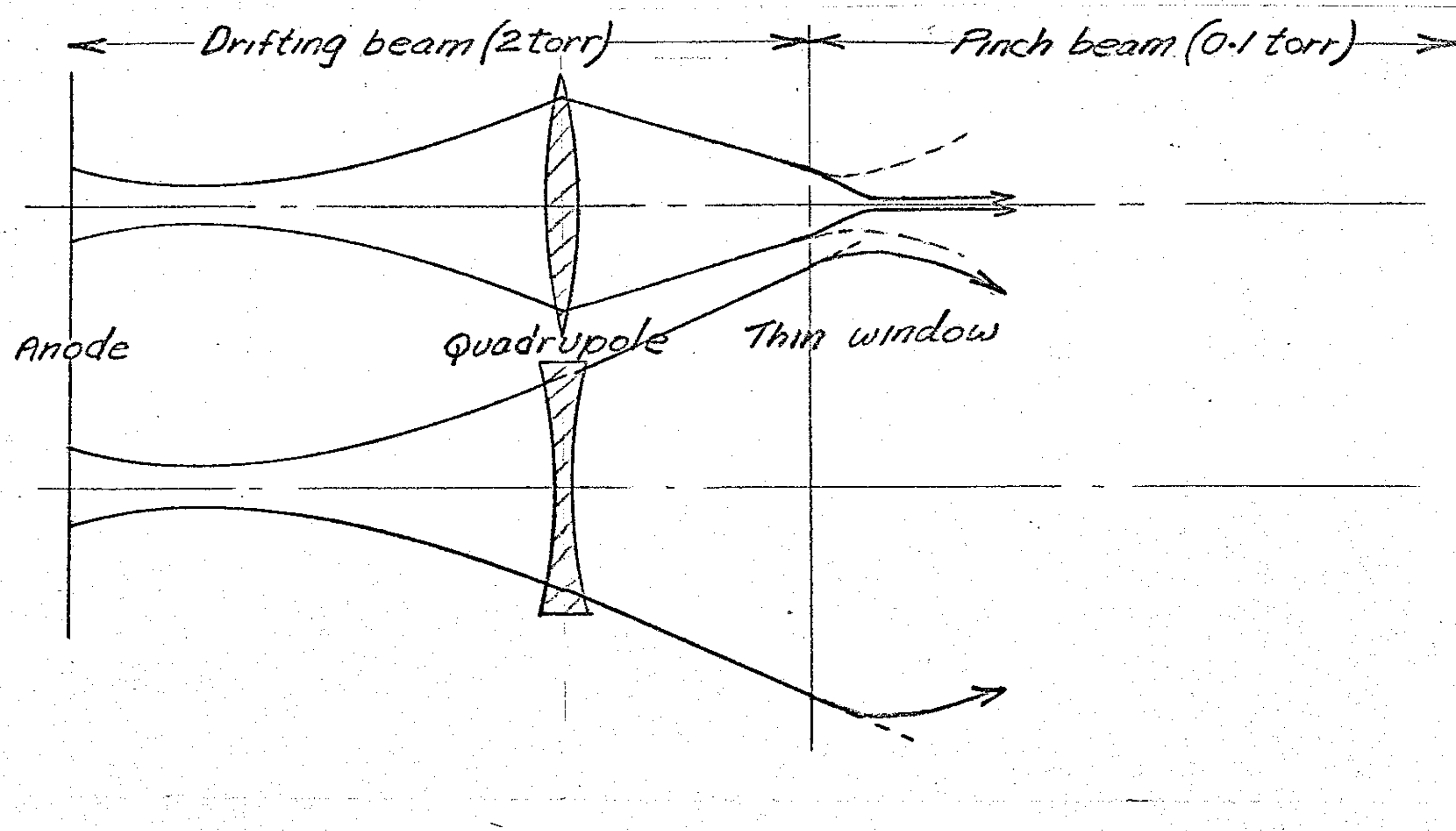


FIGURE 15. Focus followed by pinch.

The prospects of analytical and experimental work on this problem are exciting. This report is an attempt to bring the "real world" into an analysis, i.e., finite phase space area and energy spread are included. The real world however must be looked at experimentally and a very brief series of experiments reported in PIIR-30-67 by L. Hatch has shown that drifting electron beams are indeed distorted by a quadrupole magnet.

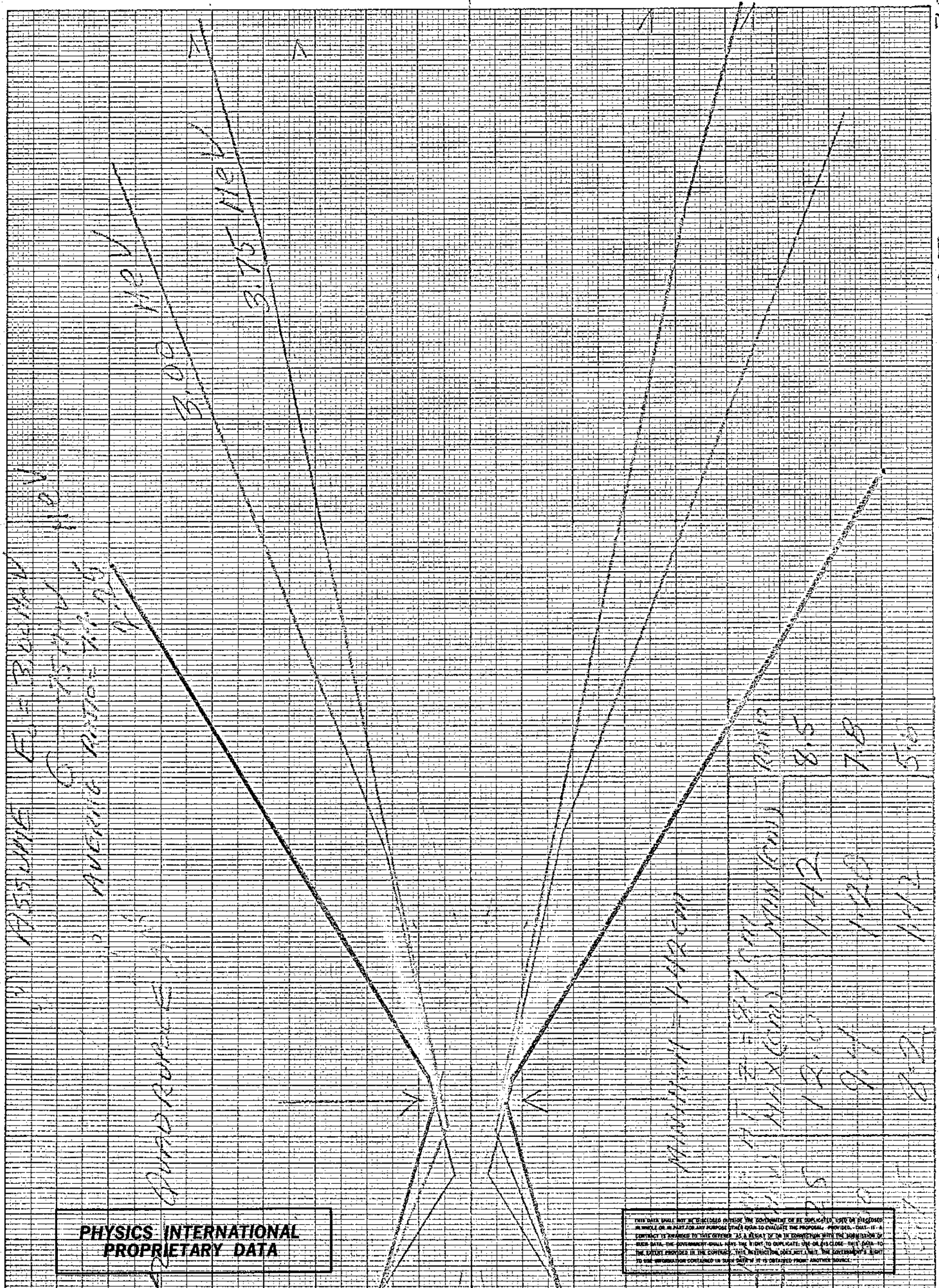


FIGURE 7

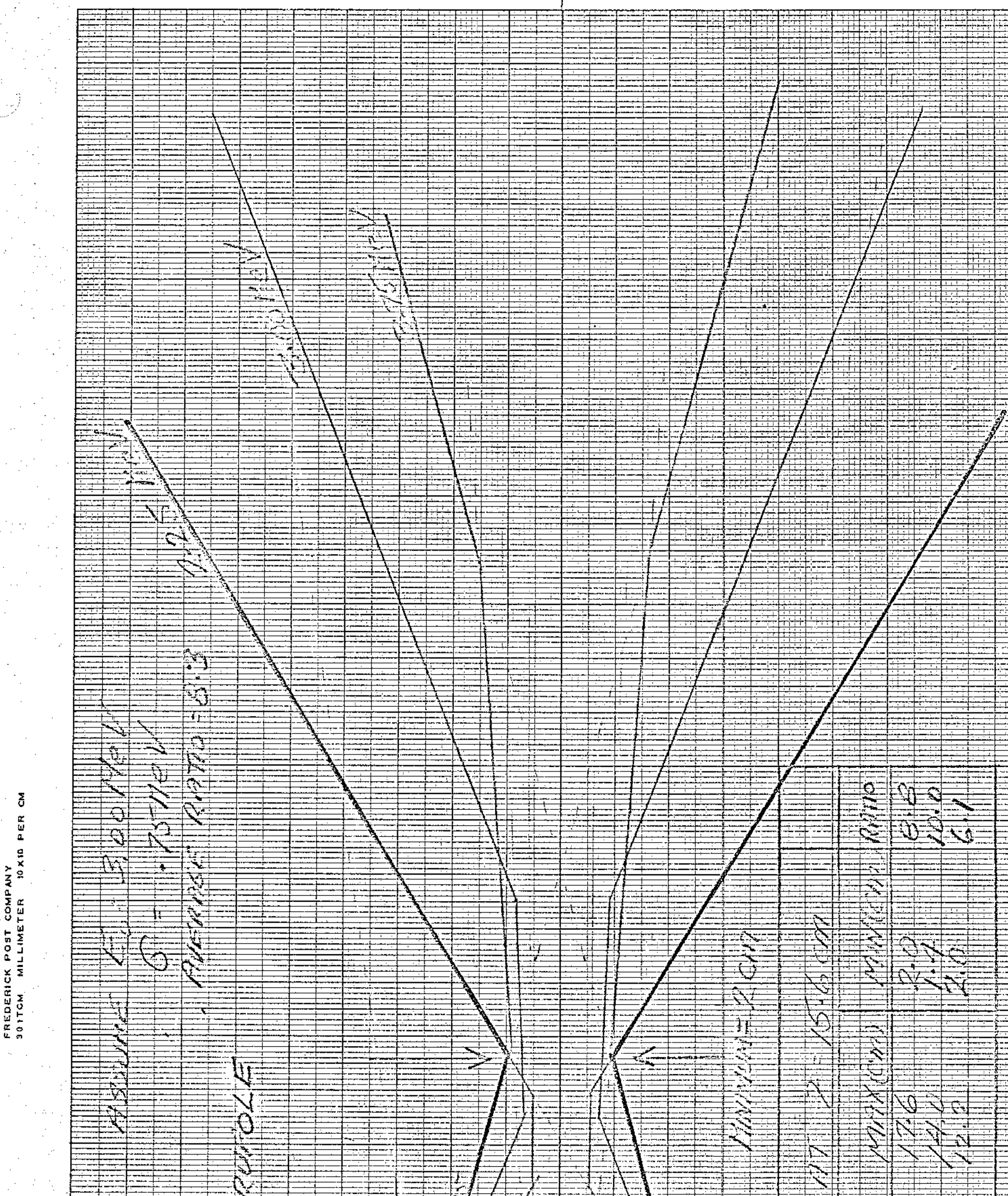

FREDERICK POST COMPANY
301TCM MILLIMETER 10 X 10 PER CM
PHYSICS INTERNATIONAL PROPRIETARY DATA
QUADRUPOLE
AVERAGE RATIO = 8.3
MINIMUM = 2 cm
MAX (cm)
MIN (cm)
RATIO
17.6
14.0
12.2
2.0
1.4
2.0
8.8
10.0
6.1
20
30

FIGURE 8

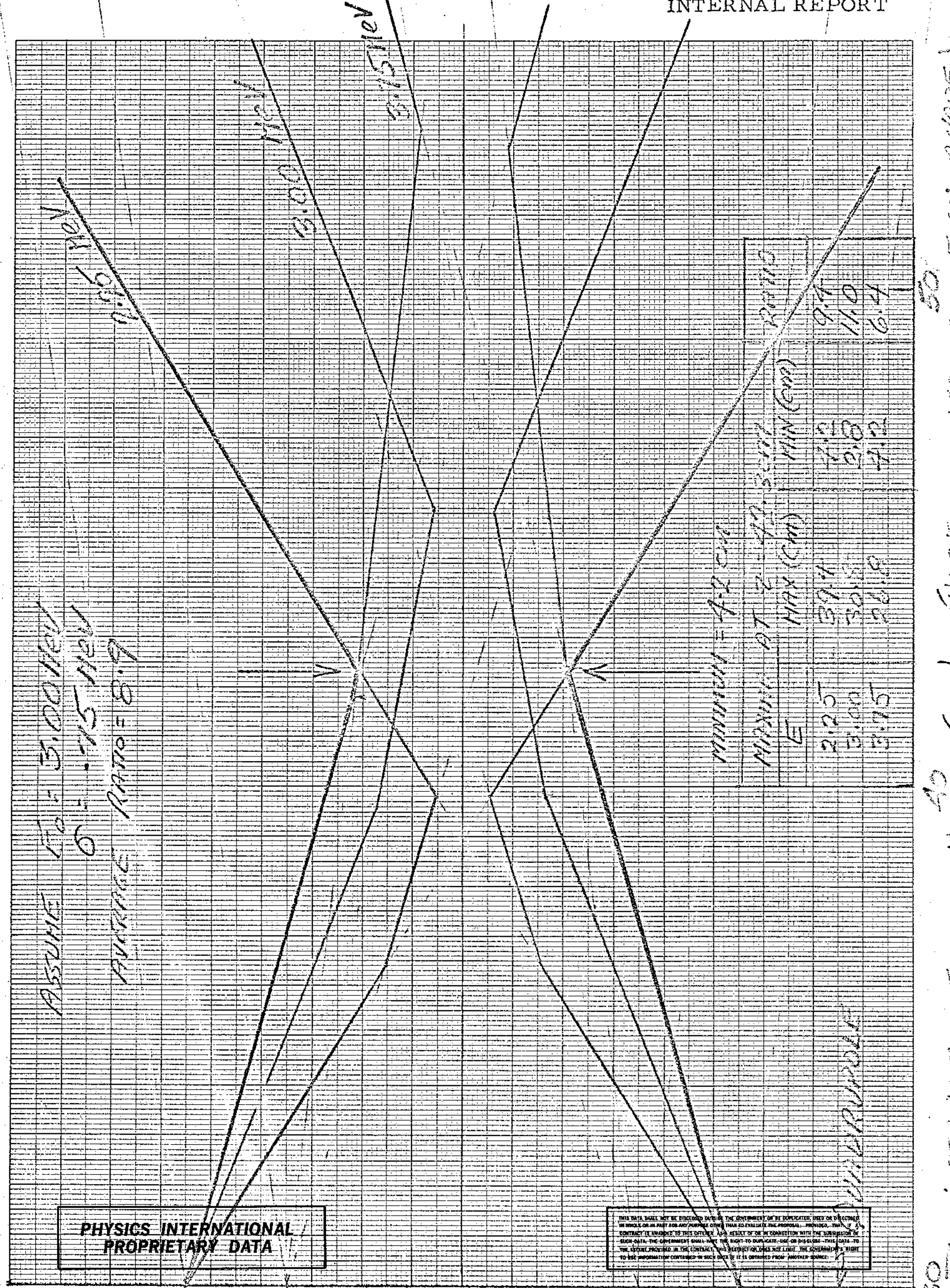

ASSUME E0 = 3.00 MeV
σ = .15 MeV
AVERAGE RATIO = 8.9
2.85 MeV
3.00 MeV
3.15 MeV
MINIMUM = 4.2 cm
E
2.85
3.00
3.15
MAX (cm)
39.4
30.8
26.9
MIN (cm)
4.2
2.8
4.2
RATIO
9.4
11.0
6.4
QUADRUPOLE
PHYSICS INTERNATIONAL PROPRIETARY DATA
FREDERICK POST COMPANY 301TGM MILLIMETER 10 X 10 PER CM

FIGURE 9

SCALE-FULL SIZE

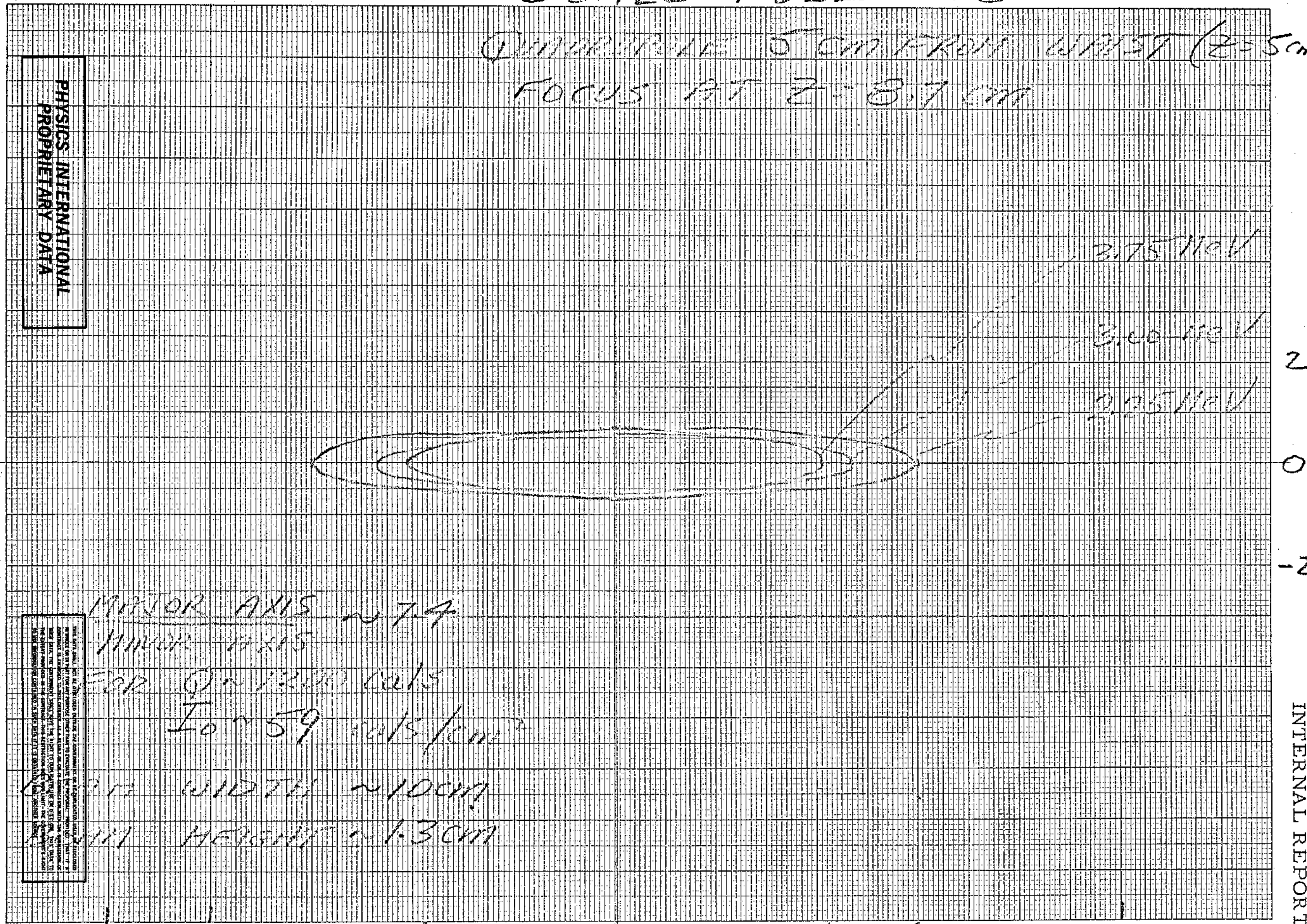
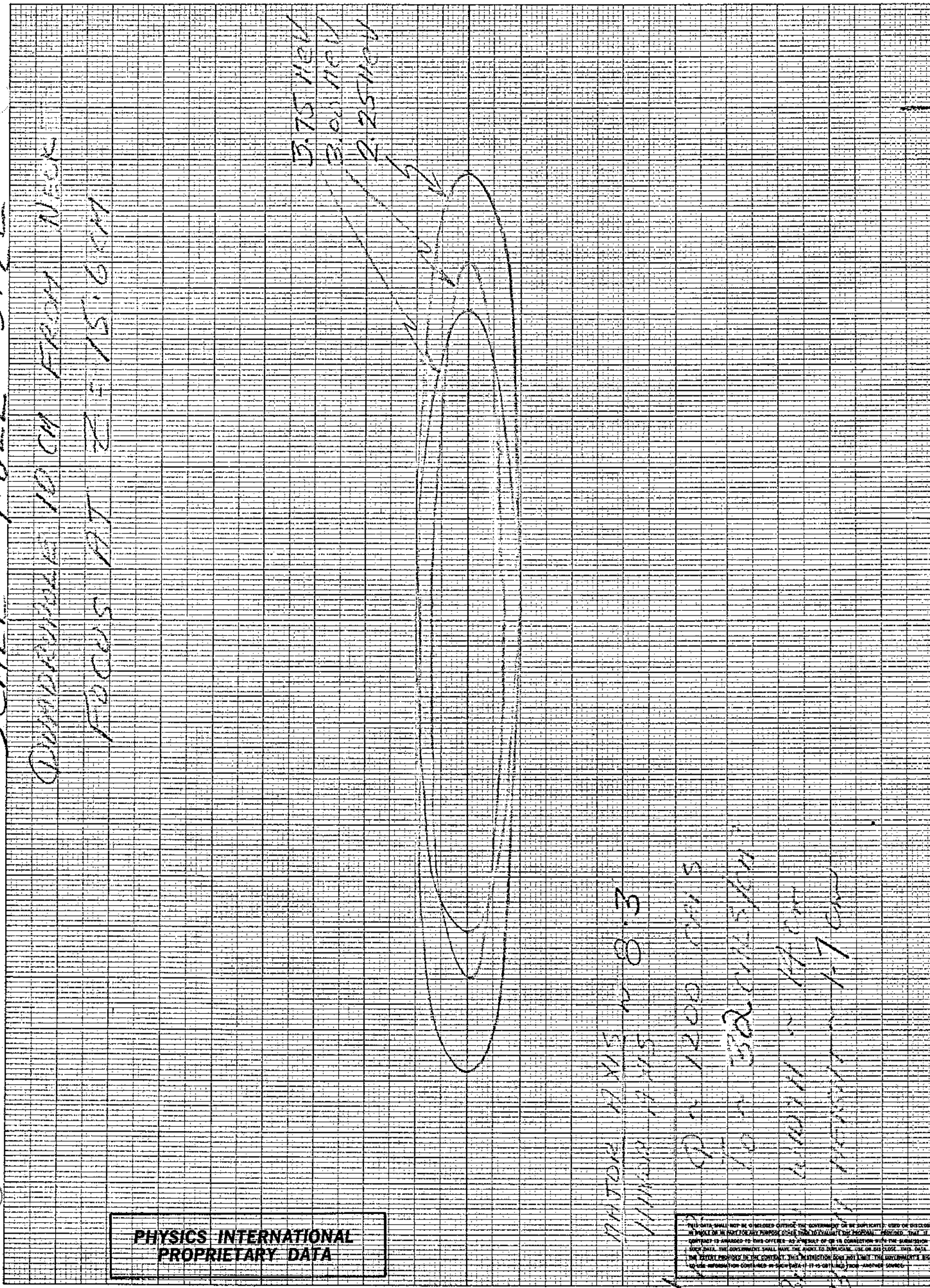
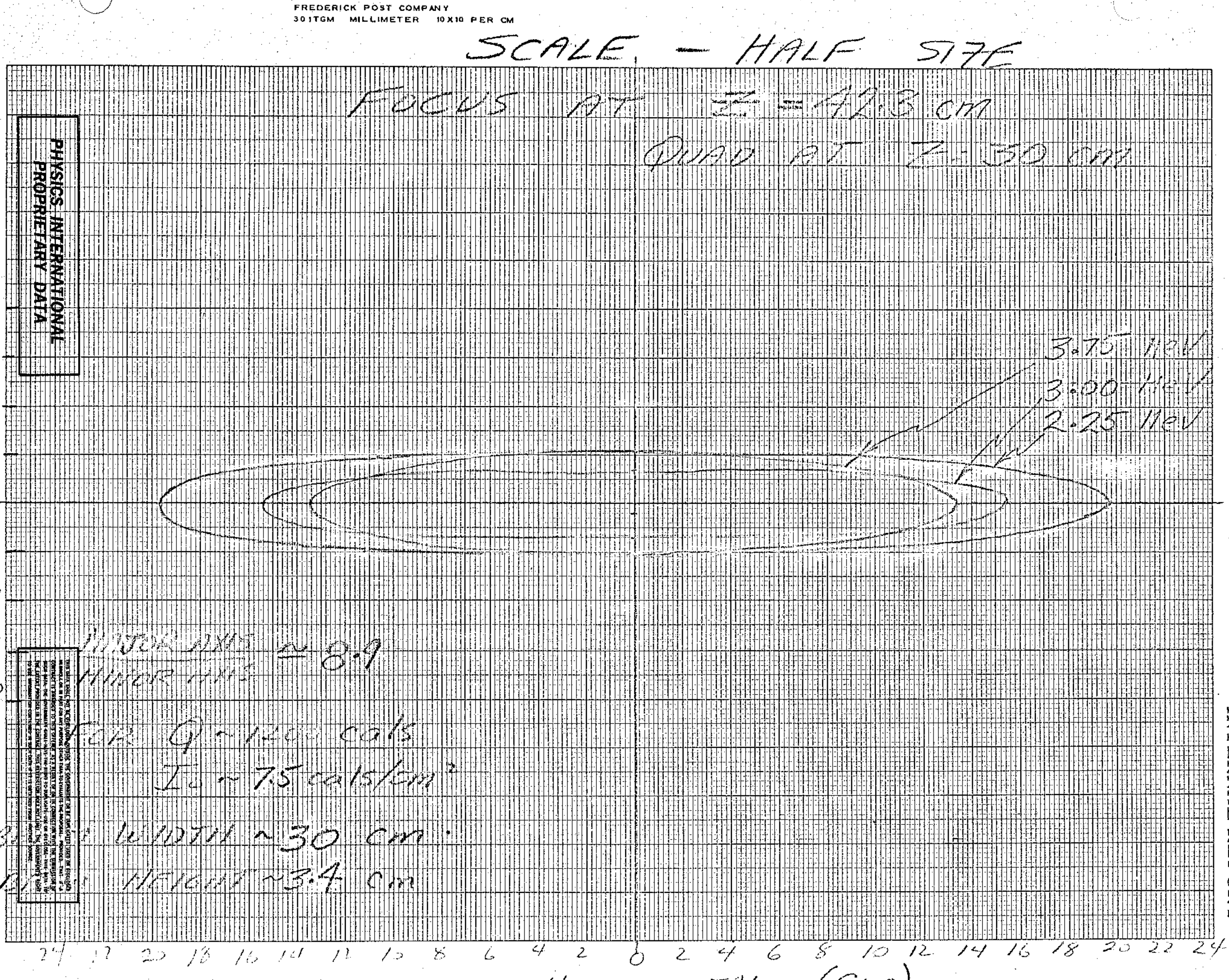


FIGURE 10



SCALE - FULL SIZE
QUADRUPOLE 10 CM FROM NECK
FOCUS AT Z = 15.6 CM
3.75 MeV
3.00 MeV
2.25 MeV

FIGURE 11



VERTICAL (cm) FIGURE 12

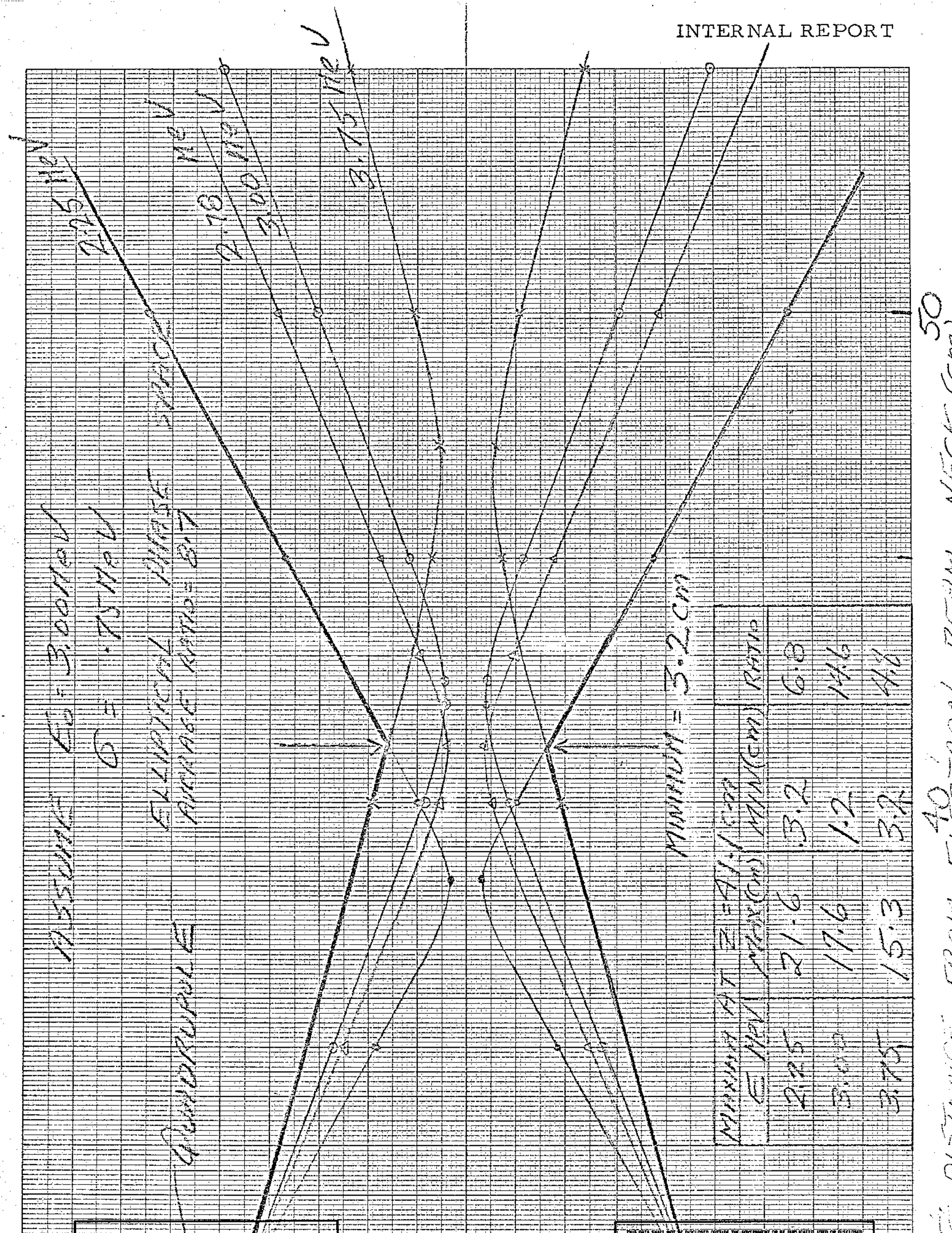

ASSUME E0 = 3.00 MeV
δ = .75 MeV
QUADRUPOLE ELLIPTICAL PHASE SPACE
AVERAGE RATIO = 8.7
MINIMUM = 3.2 cm
2.25 MeV
2.78 MeV
3.00 MeV
3.75 MeV
MAXIMUM AT Z = 41 cm
E MeV | MAX (cm) | MIN (cm) | RATIO
2.25 | 21.6 | 3.2 | 6.8
3.00 | 17.6 | 1.2 | 14.6
3.75 | 15.3 | 3.2 | 4.8
30
40
50
PHYSICS INTERNATIONAL PROPRIETARY DATA
FREDERICK POST COMPANY 301TGM MILLIMETER 10 X 10 PER CM

FIGURE 13

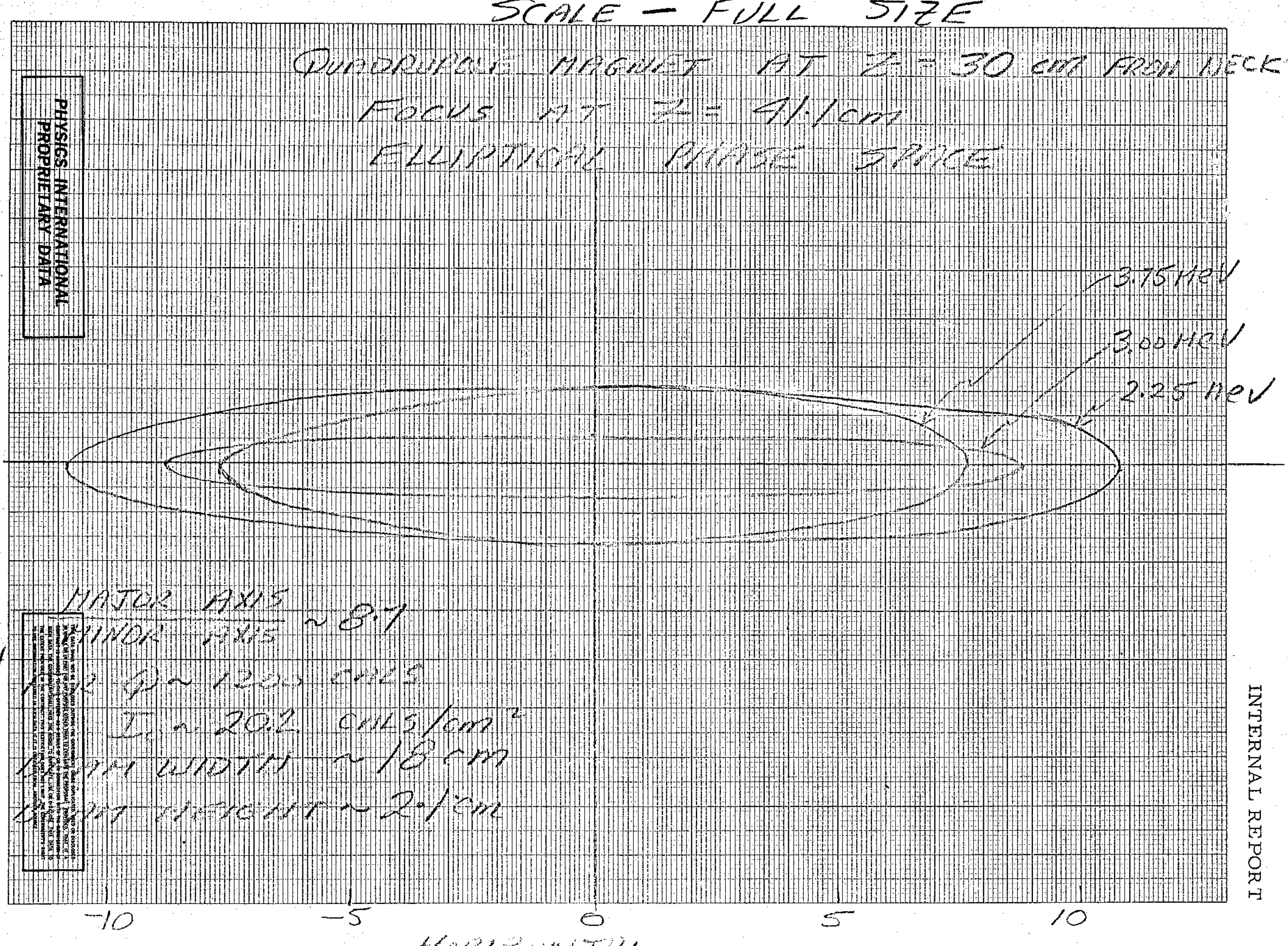

SCALE — FULL SIZE
QUADRUPOLE MAGNET AT Z = 30 cm FROM NECK
FOCUS AT Z = 41.1 cm
ELLIPTICAL PHASE SPACE
3.75 MeV
3.00 MeV
2.25 MeV
MAJOR AXIS / MINOR AXIS ~ 8.7
Q ~ 1200 CMLS
I ~ 20.2 CMLS/cm²
BEAM WIDTH ~ 18 cm
BEAM HEIGHT ~ 2.1 cm
PHYSICS INTERNATIONAL
PROPRIETARY DATA
-10
-5
0
5
10
4
3
2
1
0
-1
-2
-3
-4

FIGURE 14.